Positivity of small ball probabilities of a Gaussian random field, and its applications to random Schrödinger operators

https://www.math.h.kyoto-u.ac.jp/users/ueki/spec5.pdf
https://www.math.h.kyoto-u.ac.jp/users/ueki/presen1 spec-paper

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March 18, 2024

Spectrum of a random Schrödinger operator

$(X^\omega(x))_{x\in\mathbb{R}^d}$: a real Gaussian random field on $\mathbb{R}^d$

$\mathbb{E}[X^\omega(x)] = 0$, $\mathbb{E}[X^\omega(x)X^\omega(y)] = \gamma(x-y)$

(A1) $\gamma(0) > 0$, $\lim_{|x|\to 0} \gamma(x) = 0$, and γ is integrable and Hölder continuous at 0.

Theorem (Spectrum)

$\Rightarrow$ spec$(-\Delta + X^\omega$ *on* $L^2(\mathbb{R}^d)) = \mathbb{R}$.

Pastur, L. and Figotin, A., Spectra of random and almost-periodic operators, Springer, Berlin, 1992.

Proof based on

(P1) $$\mathbb{P}\big(\sup_{|x|\le\ell} |X^\omega(x) - \lambda| < \eta\big) > 0 \text{ for } \forall \ell, \eta > 0 \text{ and } \lambda \in \mathbb{R}.$$

Small ball probability

Li, W. V. and Shao, Q.-M., Gaussian processes: inequalities, small ball probabilities and applications, Stochastic processes: theory and methods, Handbook of Statist., 19, North-Holland, Amsterdam, 2001, 533—597.

Theorem (Talagrand (1993), Ledoux (1996))

$$\mathbb{P}(\sup_{|x|\le\ell} |X^\omega(x)| < \eta) \ge \exp(-\underbrace{N(\eta,\ell)}_{\textit{entropy number}})$$

Proof based on correlation inequality by Khatri (1967), Sidak (1967, 1968)

$$\mathbb{P}\Big(|X^\omega(x_1)| \vee \max_{2\le i\le n} |X^\omega(x_i)| < \eta\Big) \ge \mathbb{P}\Big(|X^\omega(x_1)| < \eta\Big)\mathbb{P}\Big(\max_{2\le i\le n} |X^\omega(x_i)| < \eta\Big)$$

Shifted small ball

We use the Cameron-Martin theorem

$$\mathbb{P}(X^\omega - h \in E) = \int_E \exp\Big(-\frac{\|h\|_H^2}{2} - (X,h)_H\Big)\mathbb{P}(X^\omega \in dX)$$

for any $h \in H$, where $\|\cdot\|_H$ is the norm of the Cameron-Martin space H and $(\cdot,\cdot)_H$ is the stochastic extension of the inner product of the space:

Lemma (Hoffmann-Jorgensen, Shepp, Dudley (1979), de Acosta (1983))

$$\mathbb{P}(\sup_{|x|\le\ell} |X^\omega(x) - h(x)| < \eta) \ge \exp(-\|h\|_H)\mathbb{P}(\sup_{|x|\le\ell} |X^\omega(x)| < \eta).$$

The norm $\|\cdot\|_H$ appears formally as $\mathbb{P}(X^\omega \in E) = \int_E \exp\Big(-\frac{\|X\|_H^2}{2}\Big)\frac{DX}{Z}$

In our case, $\|h\|_H = \Big(\int_{\mathbb{R}^d} \frac{|\widehat{h}(\zeta)|^2}{\widehat{\gamma}(\zeta)} d\zeta\Big)^{1/2}$.

Sufficient conditions for (P1)

Theorem

Under (A1),

(P1) $\mathbb{P}(\sup_{|x|\le\ell} |X^\omega(x) - \lambda| < \eta) > 0$ *for* $\forall \ell, \eta > 0$ *and* $\lambda \in \mathbb{R}$.

$\Uparrow$

(A3) For any $\ell, \eta > 0$, *there exists a function* h *on* $\mathbb{R}^d$ *such that*

$$\sup_{|x|\le\ell} |1 - h(x)| < \eta \text{ and } \int_{\mathbb{R}^d} \frac{|\widehat{h}(\zeta)|^2}{\widehat{\gamma}(\zeta)} d\zeta < \infty$$

$\Uparrow$

(A2) $\int_{|\zeta|<\delta} \widehat{\gamma}(\zeta) d\zeta > 0$ *for any* $\delta > 0$

The small ball probability for the Random Schrödinger operators

Theorem

(A1)

$\Downarrow$ *straightforward*

(P2) $\exists V : \mathbb{R}^d \to \mathbb{R}$: *rapidly decreasing,* $\not\equiv 0$ *s.t.*

$\mathbb{P}(\sup_{|x|\le\ell} |X^\omega(x) - \lambda V(x)| < \eta) > 0$ *for* $\forall \ell, \eta > 0$ *and* $\lambda \in \mathbb{R}$.

$\Downarrow$

$\mathrm{spec}(-\Delta + X^\omega$ *on* $L^2(\mathbb{R}^d)) = \mathbb{R}$

$\because$ For $\forall \mu < 0$, $\exists \lambda_\mu \in \mathbb{R}$ s.t. $\inf \mathrm{spec}(-\Delta + \lambda_\mu V) = \mu$

By Weyl's criterion, $\exists \{\varphi_n\}_n \subset C_0^\infty(\mathbb{R}^d)$ s.t. $\|\varphi_n\|_{L^2} = 1$,

$\|(-\Delta + \lambda_\mu V - \mu)\varphi_n\|_{L^2} \le \varepsilon_n \xrightarrow{n\to\infty} 0$

Under the event $\sup_{\mathrm{supp}\,\varphi_n} |X^\omega(x) - \lambda_\mu V(x)| < \eta$, we have

$\|(-\Delta + X^\omega - \mu)\varphi_n\|_{L^2} \le \eta + \varepsilon_n$

Remarks for the proof

$[0, \infty) \subset \mathrm{spec}(-\Delta + X^\omega)$ is proven
only by the positivity of non shifted small ball probability

$$\mathbb{P}(\sup_{|x|\le \ell} |X^\omega(x) - V(x)| < \eta) > 0$$

$\Downarrow$ Ergodicity

$$\mathbb{P}(\sup_{|x-x_0|\le \ell} |X^\omega(x) - V(x)| < \eta \text{ for some } x_0 \in \mathbb{R}^d) = 1$$

cf. Ando, K., Iwatsuka, A., Kaminaga, M. and Nakano, F., (2006)
Similar discussions for the Poisson type random potential including a counter example

Random magnetic field+Uniform magnetic field

Theorem

$\mathrm{spec}((i\nabla + A^\omega)^2$ *on* $L^2(\mathbb{R}^2)) = [0,\infty)$, *where* $\nabla \times A^\omega = X^\omega - \lambda$ *with* $\lambda \in \mathbb{R}$

$\Uparrow$

(P1)' $\mathbb{P}(\sup_{|x|\le \ell} |X^\omega(x) - \lambda| < \eta) > 0$ *for* $\forall \ell, \eta > 0$

$\Uparrow$ *This is already shown.*

(A1) and

(A2) $\displaystyle\int_{|\zeta|<\delta} \widehat{\gamma}(\zeta) d\zeta > 0$ *for any* $\delta > 0$

$\because$ For $\forall \mu > 0$, take $\exists \{\varphi_n\}_n \subset C_0^\infty(\mathbb{R}^2)$ s.t. $\|\varphi_n\|_{L^2} = 1$,

$\|(-\Delta - \mu)\varphi_n\|_{L^2} \le \varepsilon_n \overset{n\to\infty}{\longrightarrow} 0$

Under the event $\sup_{\mathrm{supp}\,\varphi_n} |X^\omega(x) - \lambda| < \eta$, we can take a small A^ω so that

$\|((i\nabla + A^\omega)^2 - \mu)\varphi_n\|_{L^2} \le c_n \eta + \varepsilon_n$

Anderson localization under a Gaussian random magnetic field

Ueki, N., Wegner estimates, Lifshitz tails and Anderson localization for Gaussian random magnetic fields, J. Math. Phys., **57**(7) (2016), 071502.

Technical conditions including

(A2)' $\lim_{\varepsilon \downarrow 0} \sup_{\mathcal{R} \in [1,\infty)} |\{\zeta \in \mathbb{R}^2 : |\zeta| \le \mathcal{R}, \widehat{\gamma}(\zeta)(1+|\zeta|)^m \le \varepsilon\}|/(\mathcal{R}^2 \varepsilon^\mu) = 0$

for some $\mu \in (0,\infty)$ and $m \in (8,\infty)$ (stronger than (A2))

$\nabla \times A^\omega = X^\omega - \lambda$ with some $\lambda \in \mathbb{R}$

$\Rightarrow \exists E_0 \in (0,\infty)$ s.t.

spec($(i\nabla + A^\omega)^2$ on $L^2(\mathbb{R}^2)) \cap [0, E_0]$ is pure point

and the corresponding eigenfunctions decay exponentially.